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L^∞ Estimates for the Space-Homogeneous Boltzmann Equation

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This paper studies the boundedness of solutions f of the initial-value problem for the space-homogeneous Boltzmann equation for inverse kth power forces, when $k > 5$, and under angular cutoff. The main result is that if the initial value is $f_0 \geqslant 0$ with $(1 + |v|^2)f_0 \in L^1$, and $(1 + |v|)^s f_0 \in L^\infty$ for some $s > 2$, then $(1 + |v|)^{s'} f_t \in L^\infty$ for $t > 0$ and $\mathrm{ess}_{v,t}\sup(1 + |v|)^{s'} f(v, t) < \infty$ for any $s' \leqslant s$ when $s \leqslant 5$, and any $s' < s$ if $s > 5$.

KEY WORDS: Nonlinear Boltzmann equation; space-homogeneous Boltzmann equation; L^∞ Boltzmann solutions.

1. INTRODUCTION

This paper studies the boundedness of solutions f of the initial-value problem for the space-homogeneous Boltzmann equation with inverse kth power forces, when $k > 5$, and under angular cutoff. The main result, contained in Theorem 2 below, is that if the initial value is $f_0 \geqslant 0$ with $(1 + |v|^2)f_0 \in L^1$, and $(1 + |v|)^s f_0 \in L^\infty$ for some $s > 2$, then $(1 + |v|)^{s'} f_t \in L^\infty$ for $t > 0$, and $\mathrm{ess}_{v,t}\sup(1 + |v|)^{s'} f(v, t) < \infty$ for any $s' \leqslant s$ when $s \leqslant 5$, and any $s' < s$ if $s > 5$.

The only previous results in this direction that we are aware of are by Carleman[2] for elastic collisions and $s > 6$. For a comment see the remark after Theorem 2. Our proof is based on a sharpening of his methods coupled with the use of the by now fairly well developed L^1 theory.

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2. PRELIMINARIES AND STATEMENT OF THE MAIN THEOREM

In this paper we study spatially homogeneous solutions of the nonlinear Boltzmann equation when there are no exterior forces, i.e., solutions of the equation

$$D_t f(v,t) = Qf(v,t) \qquad (t>0, v \in R^3) \tag{1}$$

with Cauchy data

$$f(v,0) = f_0(v) \geqslant 0 \qquad (v \in R^3)$$

Here Q denotes the collision operator

$$\begin{gathered} Qf(v_1) = \int_{R^3 \times B} [f \otimes f(\mathscr{J}_u(v_1,v_2)) - f \otimes f(v_1,v_2)] \bar{S}(u,w)\, d\mu_u\, dv_2, \\ f \otimes g(v_1,v_2) = f(v_1) g(v_2) \end{gathered} \tag{2}$$

and

$$w(v_1,v_2) = |v_1 - v_2|$$

For molecules with angular cutoff the impact parameter u is restricted to a set

$$B = B(\epsilon) = \{u = (\theta,\varphi) \in R^2;\ 0 \leqslant \theta \leqslant \pi/2 - \epsilon,\ 0 \leqslant \varphi \leqslant 2\pi\},$$

and $\bar{S}$ is independent of φ,

$$\bar{S}(u,w) = S(\theta,w), \qquad d\mu_u = \sin\theta\, d\theta\, d\varphi$$

$\mathscr{J}$ denotes a diffeomorphism of the velocity and impact parameters. If the impact parameter is u, the asymptotic velocities v_1', v_2' after collision of two colliding particles with initial velocities v_1 and v_2 are given by

$$(v_1', v_2') = \mathscr{J}_u(v_1,v_2)$$

Set

$$\begin{aligned} &p: R^3 \times R^3 \to R^3 && (v_1,v_2) \to v_1 + v_2 \\ &T: R^3 \times R^3 \to R && (v_1,v_2) \to |v_1|^2 + |v_2|^2 \\ &\Sigma: R^3 \times R^3 \to R^3 \times R^3 && (v_1,v_2) \to (v_2,v_1) \end{aligned}$$

On physical grounds $\mathscr{J}$ is subject to the restrictions

$$\begin{gathered} p \circ \mathscr{J}_u = p, \qquad T \circ \mathscr{J}_u = T, \\ \Sigma \circ \mathscr{J}_u = \mathscr{J}_u \circ \Sigma, \qquad \mathscr{J} \circ \mathscr{J} = \text{identity} \end{gathered} \tag{3}$$

It follows that

$$w \circ \Sigma = w \circ \mathscr{J}_u = w \tag{4}$$

For a thorough discussion of (2)–(4), see Ref. 4.

We shall also introduce the familiar decomposition of Qf,

$$Qf(v_1) = Jf(v_1) - f(v_1)Lf(v_1)$$

with

$$Lf(v_1) = \int_{R^3 \times B} f(v_2)\bar{S}(u, w)\, d\mu_u\, dv_2$$

and

$$Jf(v_1) = \int_{R^3 \times B} f \otimes f(v_1', v_2')\bar{S}(u, w)\, d\mu_u\, dv_2$$

together with

$$Jfg(v_1) = \int_{R^3 \times B} f \otimes g(v_1', v_2')\bar{S}(u, w)\, d\mu_u\, dv_2$$

The following norms will be used:

$$\|f\|_{1,s} = \int_{R^3} |f(v)|(1 + |v|^2)^{s/2}\, dv \qquad (s \geqslant 0)$$

$$\|f\|_{\infty,s} = \operatorname*{ess\,sup}_{v \in R^3} |f(v)|(1 + |v|)^s \qquad (s \geqslant 0)$$

as well as the corresponding weighted spaces

$$L_s^p = L_s^p(R^3) = \{f;\, f \text{ measurable on } R^3,\ \|f\|_{p,s} < \infty\} \qquad (s \geqslant 0, p = 1, \infty)$$

and their positive cones

$$L_s^{p,+} = \{f \in L_s^p;\, f \geqslant 0 \text{ a.e.}\}$$

Set

$$R_+ = \{t \in R;\, t \geqslant 0\}$$

and define

$$\|f\|_{p,s,+} = \sup_{t > 0} \|f(\cdot, t)\|_{p,s}$$

for functions

$$f: R_+ \to L_s^p$$

A.o. for inverse kth power forces, $5 < k < \infty$, the weight function S factorizes (cf. Ref. 4, p. 181) as

$$S(\theta, w) = S_k(\theta, w) = h_k(\theta)w^\beta = h(\theta)w^\beta \tag{5}$$

with

$$0 < \beta = (k-5)/(k-1) < 1$$

The dependence of S and h on k will usually be suppressed below. We shall also refer to the case of elastic collisions with

$$S(\theta, w) = h(\theta) w, \qquad h(\theta) = \cos\theta$$

as $S_\infty(\theta, w)$. In the present paper we only consider

$$S_k(\theta, w), \qquad 5 < k \leqslant \infty$$

with angular cutoff at

$$\theta = \pi/2 - \epsilon$$

taking

$$\begin{aligned} \epsilon > 0 \qquad & \text{for} \quad 5 < k < \infty \\ \epsilon = 0 \qquad & \text{for} \quad k = \infty \end{aligned}$$

It is well known that h is bounded under these cutoffs. For a proof see, e.g., Ref. 4, pp. 181 and 317.

In the following lemma we summarize some essentially well known properties of the Boltzmann equation (1) in the L^1-case.

Lemma 1. Let f_0 be given with

$$f_0 \in L_2^{1,+}, \qquad f_0 \log f_0 \in L_0^1 \tag{6}$$

Then for inverse kth power forces with $5 < k \leqslant \infty$, and angular cutoffs, there is a solution

$$f : R_+ \to L_2^{1,+}$$

of the Boltzmann equation (1) with initial value

$$f(v, 0) = f_0(v)$$

such that

$$\int_{R^3} f(v,t)\,dv = \int_{R^3} f_0(v)\,dv, \qquad \int_{R^3} v f(v,t)\,dv = \int_{R^3} v f_0(v)\,dv \tag{7}$$

$$\int_{R^3} f(v,t)|v|^2\,dv \leqslant \int_{R^3} f_0(v)|v|^2\,dv \tag{8}$$

$$\int_{R^3} f(v,t) \log f(v,t)\,dv \leqslant \int_{R^3} f_0(v) \log f_0(v)\,dv \tag{9}$$

If $\|f_0\|_{1,s_1} < \infty$ for some $s_1 > 2$, then the solution can be chosen so that

$$\|f\|_{1,s_1,+} < \infty \tag{10}$$

3. SKETCH OF PROOF

The existence results including (7)–(9) follow by the methods of Ref. 1. The global estimates of higher moments in that paper are not strong enough to give (10). They have been improved by Elmroth in Ref. 3 for the case of kth power molecules without cutoff. His estimates can also be used in the present cutoff case to prove (10).

Remark. Any solution of Lemma 1 is for a.e. v_1 a continuously differentiable function

$$f(v, \cdot) : R_+ \to R_+$$

satisfying the Boltzmann equation (1) pointwise. For a discussion see, e.g. Ref. 4, Chap. XXI.

The main result of this paper is an L^∞ analog of (10) contained in the following theorem, and proved in the final section.

Theorem 2. Suppose

$$f_0 \in L_2^{1,+} \cap L_{s_2}^\infty \tag{11}$$

for some $s_2 > 2$, and f is any solution of the Boltzmann equation (1) with initial value f_0, satisfying (7)–(9), and if $s_2 > 5$ (10) for all $s_1 < s_2 - 3$. Then f is a mapping from R_+ to $L_{\bar{s}_2}^\infty$, and

$$\|f\|_{\infty,\bar{s}_2,+} \leqslant C_k^0(\bar{s}_2) < \infty \tag{12}$$

for any $\bar{s}_2 \leqslant s_2$ when $s_2 \leqslant 5$, and any $\bar{s}_2 < s_2$, when $s_2 > 5$.

Remark. Carleman's L^∞ estimates in Ref. 2 correspond in our theorem to the case $k = \infty$, $s_2 > 6$. There he also obtains (12) for $\bar{s}_2 = s_2$.

In Theorem 2 it follows from the hypothesis $f_0 \in L_0^\infty \cap L_0^1$ that $f_0 \log f_0 \in L_0^1$. And so (6) holds, implying the existence of at least one solution f satisfying (7)–(9). If $s_2 > 5$, then moreover $f_0 \in L_{s_1}^{1,+}$ for any $s_1 < s_2 - 3$, and the solution can be so chosen that (10) holds for these values of s_1.

Constants are in this paper denoted by C, and C_k denotes any constant only depending on k and the relevant L_s^p space. A constant also depending on f_0 is denoted by C_k^0. Other kind of dependence, when emphasized, will be indicated by brackets as the dependence on $\bar{s}_2$ in $C_k^0(\bar{s}_2)$ above.

For the proof of Theorem 2 we shall repeatedly use the following well-known lemma:

Lemma 3. Suppose h_1 and h_2 are continuous, real-valued functions on R_+ with $h_1 > 0$. If

$$f' + h_1 f \leqslant h_2 \qquad (t > 0)$$

then

$$\sup_{t>0} f(t) \leqslant \max\Big(f(0), \sup_{t>0} h_2(t)/h_1(t)\Big)$$

We shall also need some transformations of Jf and Lf. Let $E_{v,\bar{v}}$ denote the plane in R^3 through v and orthogonal to $v-\bar{v}$,

$$E_{v,\bar{v}}=\{v_1\in R^3;(v-\bar{v})(v-v_1)=0\}$$

Its Lebesgue measure is denoted by dE_1. By elementary computations we can express Jfg through integration over such planes,

$$\begin{aligned} Jfg(v_1)&=\int_{R^3}\int_0^{2\pi}\int_0^{\pi/2-\epsilon}f(v_1')g(v_2')w\beta(v_1',v_2')h(\theta)\sin\theta\,d\theta\,d\varphi\,dv_2\\ &=\int_{R^3}f(v_1')\int_{E_{v_1,v_1',\epsilon}}g(v_2')h(\theta)\cos^{-2}\theta\,w^{-\alpha}(v_1',v_2')\,dE_2'\,dv_1' \end{aligned}\tag{13}$$

Here

$$\alpha=2-(k-5)/(k-1)=(k+3)/(k-1)$$

and $E_{v_1,v_1',\epsilon}$ is the subset of the plane $E_{v_1,v_1'}$ corresponding to an angular cutoff at $\theta=\pi/2-\epsilon$. For a proof see Ref. 2, p. 32.

Below we shall often use "for a.e. $E_{v,\bar{v}}$" in the sense that for each v the property in question holds in a.e. direction $(v-\bar{v})/|v-\bar{v}|$.

The integral

$$\int_{R^3}\phi(v_1)Jfg(v_1)\,dv_1$$

can be transformed as follows:

$$\begin{aligned} \int_{R^3}\phi(v_1)Jfg(v_1)\,dv_1&=\int_{R^3\times R^3}\phi(v_1)\int_0^{2\pi}\int_0^{\pi/2-\epsilon}f(v_1')g(v_2')h(\theta)\\ &\qquad\times\sin\theta w^{\beta}(v_1,v_2)\,d\theta\,d\varphi\,dv_1\,dv_2\\ &=\int_{R^3\times R^3}f(v_1)g(v_2)\int_{\rho_\epsilon}h(\theta)\\ &\qquad\times\cos^{-1}\theta\,w^{-\alpha}(v_1,v_2)\phi(v_1')\,d\sigma'\,dv_1\,dv_2 \end{aligned}\tag{14}$$

Here ρ_ϵ is the cutoff at $\theta=\pi/2-\epsilon$ of the sphere ρ with center $(v_1+v_2)/2$, radius $|v_1-v_2|/2$, and θ the angle between $v_1'-v_1$ and v_2-v_1. The measure $d\sigma'$ is the Lebesgue measure on ρ_ϵ. For a proof see Ref. 2, p. 33.

Recalling (5) we can write Lf as

$$\begin{aligned} Lf(v_1)&=\int_{R^3}\int_0^{2\pi}\int_0^{\pi/2-\epsilon}f(v_2)w^{\beta}(v_1,v_2)h(\theta)\sin\theta\,d\theta\,d\varphi\,dv_2\\ &=2\pi\int_0^{\pi/2-\epsilon}h(\theta)\sin\theta\,d\theta\int_{R^3}f(v_2)w^{\beta}(v_1,v_2)\,dv_2\\ &=C_k\int_{R^3}f(v_2)w^{\beta}(v_1,v_2)\,dv_2 \end{aligned}\tag{15}$$

4. ESTIMATES OF THE COLLISION TERM

In this section we collect some estimates of Lf and of Jf and its factors, which later will be used in the proof of Theorem 2.

Lemma 4. If $f \in L_2^{1,+}$ and $\int_{R^3} f(v)\log^+ f(v)\,dv < C^0$, then

$$Lf(v) > C_k(C^0, \|f\|_{1,2}, \|f\|_{1,0})(1+|v|)^\beta$$

Proof. We notice that

$$\int_{|v_1-v_2|<r} f(v_2)\,dv_2 \leqslant \int_{|v_1-v_2|<r,\, f(v_2)<j} f(v_2)\,dv_2 + (\log j)^{-1}\int_{f(v_2)>j} f(v_2)\log^+ f(v_2)\,dv_2 \qquad (j>1)$$

For a suitable choice of r and j only depending on $\|f\|_{1,0}$ and C^0, we get

$$\int_{|v_1-v_2|<r} f(v_2)\,dv_2 < 2^{-1}\|f\|_{1,0} \qquad (v_1 \in R^3)$$

And so by (15)

$$\begin{aligned} Lf(v_1) &= C_k\int_{R^3} f(v_2)|v_1-v_2|^\beta\,dv_2 \\ &> C_k r^\beta \int_{|v_1-v_2|>r} f(v_2)\,dv_2 > C_k r^\beta \|f\|_{1,0}/2 \end{aligned} \tag{16}$$

To get another estimate of Lf we notice that

$$|v_1-v_2|^\beta \geqslant |v_1|^\beta - |v_2|^\beta$$

if $0<\beta\leqslant 1$, thus for $\beta=(k-5)/(k-1)$ and $5<k\leqslant\infty$. Together with (15) this implies

$$Lf(v_1) = C_k\int_{R^3} f(v_2)|v_1-v_2|^\beta\,dv_2 \geqslant C_k\big(|v_1|^\beta\|f\|_{1,0} - \|f\|_{1,2}\big) \tag{17}$$

The lemma follows from (16) and (17).

Lemma 5. If $f \in L_2^{1,+}$, then

$$\left\|\int_{R^3} Jf(v_1)|v_1-v|^{-\gamma}\,dv_1\right\|_{\infty,0} \leqslant C_k(\gamma)\|f\|_{1,2}^2 \qquad (0\leqslant\gamma\leqslant\beta) \tag{18}$$

$$\left\|\int_{R^3} Jf(v_1)|v_1-v|^{-\gamma}\,dv_1\right\|_{\infty,0} \leqslant C_k(\gamma)\|f\|_{1,0}\left\|\int_{R^3} f(v_1)|v_1-v|^{\beta-\gamma}\,dv_1\right\|_{\infty,0}$$

$$(\beta\leqslant\gamma<2) \tag{19}$$

Proof. By (14)

$$\int_{R^3} Jf(v_1)|v_1 - v|^{-\gamma} dv_1 = \int_{R^3\times R^3} f(v_1)f(v_2)\int_{\rho_\epsilon} h(\theta)\cos^{-1}\theta \times |v_1 - v_2|^{-\alpha}|v_1' - v|^{-\gamma} d\sigma' \, dv_1 \, dv_2$$

For $0 \leqslant \gamma < 2$

$$\int_\rho |v_1' - v|^{-\gamma} d\sigma' \leqslant C(\gamma)|v_1 - v_2|^{-\gamma+2}$$

Since $\beta = 2 - \alpha$, the above implies that

$$\int_{R^3} Jf(v_1)|v_1 - v|^{-\gamma} dv_1 \leqslant C_k(\gamma)\int_{R^3\times R^3} f(v_1)f(v_2)|v_1 - v_2|^{\beta-\gamma} dv_1 \, dv_2$$

And so (18) follows for $0 \leqslant \gamma \leqslant \beta$, and (19) for $\beta \leqslant \gamma < 2$.

Lemma 6. Suppose that

$$s_1, s_2 \geqslant 0, \quad s_2 - s_1 \leqslant 3, \quad \text{and} \quad f \in L^{1,+}_{s_1} \cap L^\infty_{s_2}$$

Then for $0 < \gamma < 3$

$$\int_{R^3} f(v_1)|v - v_1|^{-\gamma} dv_1 < C(\|f\|_{1,s_1} + \|f\|_{\infty,s_2})(1 + |v|)^{-b}$$

where

$$b = \min(\gamma, s_1 + \gamma(s_2 - s_1)/3)$$

Proof. Set

$$O_1 = \{v_1 \,;\, |v_1| \leqslant |v|/2\}$$

$$O_2 = \{v_1 \,;\, |v - v_1| \leqslant |v|^{(s_2-s_1)/3} 2^{-1}\}$$

$$O_3 = R^3 \backslash (O_1 \cup O_2)$$

Then, as is easily checked,

$$\int_{O_1} f(v_1)|v - v_1|^{-\gamma} dv_1 \leqslant C(\|f\|_{1,0} + \|f\|_{\infty,0})(1 + |v|)^{-\gamma}$$

$$\int_{O_2} f(v_1)|v - v_1|^{-\gamma} dv_1 \leqslant C(\|f\|_{1,0} + \|f\|_{\infty,s_2})(1 + |v|)^{-s_2+(3-\gamma)(s_1-s_1)/3}$$

$$\int_{O_3} f(v_1)|v - v_1|^{-\gamma} dv_1 \leqslant C(\|f\|_{1,s_1} + \|f\|_{\infty,0})(1 + |v|)^{-s_1-\gamma(s_2-s_1)/3}$$

This proves the lemma. ■

Lemma 7. If $f \in L_0^{1,+}$, then for a.e. plane $E_{v,\bar{v}}$

$$\int_{E_{v,\bar{v}}} Jf(v_1)\,dE_1 \leqslant C_k \|f\|_{1,0} \left\| \int_{R^3} f(v_2)|v_1 - v_2|^{-\alpha+1}\,dv_2 \right\|_{\infty,0}$$

Proof. Denote by d or $d(v_1)$ the distance from v_1 to the plane $E_{v,\bar{v}}$, and set

$$\phi_j(v_1) = (j\pi^{-1})^{1/2}\exp(-jd^2) \tag{20}$$

Then

$$\lim_{j\to\infty} \int_{R^3} \phi_j(v_1) Jf(v_1)\,dv_1 = \int_{E_{v,\bar{v}}} Jf(v_1)\,dE_1$$

By (14)

$$\begin{aligned}
&\int_{R^3} \phi_j(v_1) Jf(v_1)\,dv_1 \\
&= \int_{R^3\times R^3} f(v_1)f(v_2) \int_{\rho_\epsilon} h(\theta)\cos^{-1}\theta\, |v_1 - v_2|^{-\alpha} \phi_j(v_1')\,d\sigma'\,dv_1\,dv_2 \\
&\leqslant \sup_{0\leqslant\theta\leqslant\pi/2-\epsilon} h(\theta)\cos^{-1}\theta \int_{R^3\times R^3} f(v_1)f(v_2)|v_1 - v_2|^{-\alpha} \\
&\quad \times \int_\rho \phi_j(v_1')\,d\sigma'\,dv_1\,dv_2
\end{aligned}$$

In the limit for a.e. plane $E_{v,\bar{v}}$ this gives

$$\int_{E_{v,\bar{v}}} Jf(v_1)\,dE_1 \leqslant C_k \int_{R^3\times R^3} f(v_1)f(v_2)|v_1 - v_2|^{-\alpha+1}\chi\,dv_1\,dv_2 \tag{21}$$

Here $\chi = 1$ if the plane $E_{v,\bar{v}}$ intersects the sphere ρ, otherwise $\chi = 0$. But (21) implies the desired result. ∎

Lemma 8. Given $v \in R^3$, set

$$\psi(v_1) = 0 \qquad \text{for} \quad |v_1| \leqslant |v|, \qquad = 1 \quad \text{otherwise}$$

If $s_1 \geqslant 2$ and $f \in L_{s_1}^1 \cap L_0^\infty$, then for a.e. plane $E_{\tilde{v},\bar{v}}$

$$\int_{E_{\tilde{v},\bar{v}}} \psi(v_1) Jf(v_1)\,dE_1 \leqslant C_k(\|f\|_{1,s_1} + \|f\|_{\infty,0})^2(1 + |v|)^{-s_1-\alpha+1}$$

with C_k independent of the plane $E_{\tilde{v},\bar{v}}$.

Proof. As in the proof of Lemma 7 for a.e. plane $E_{\tilde{v},\bar{v}}$

$$\int_{E_{\tilde{v},\bar{v}}} \psi(v_1) Jf(v_1)\,dE_1 \leqslant C_k \int_{R^3\times R^3} f(v_1)f(v_2)|v_1 - v_2|^{-\alpha+1}\chi\,dv_1\,dv_2$$

In this case $0 \leqslant \chi \leqslant 1$ and $\chi = 0$ if both v_1 and v_2 are small enough, e.g., if

$$|v_1| \leqslant |v|/\sqrt{2} \quad \text{and} \quad |v_2| \leqslant |v|/\sqrt{2}$$

And so by Lemma 6 (and symmetry)

$$\int_{E_{\tilde v,\bar v}} \psi(v_1)Jf(v_1)\,dE_1$$
$$\leqslant C_k\int_{|v_2|\geqslant|v|/\sqrt{2}} f(v_2)\,dv_2 \sup_{|v_2|>|v|/\sqrt{2}} \int_{R^3} f(v_1)|v_1-v_2|^{-\alpha+1}\,dv_1$$
$$\leqslant C_k\|f\|_{1,s_1}(\|f\|_{1,s_1}+\|f\|_{\infty,0})(1+|v|)^{-s_1-\alpha+1}$$

5. ESTIMATES OF THE SOLUTIONS

Inserting the estimates of Jf and Lf from the previous section into the Boltzmann equation (1) with solution f as in Lemma 1, and applying Lemma 3 we shall now prove a sequence of increasingly better estimates of f, with Theorem 2 as our final result.

Lemma 9. Under (11), if $0\leqslant\gamma<2$, then

$$\left\|\int_{R^3} f(v_1,t)|v-v_1|^{-\gamma}\,dv_1\right\|_{\infty,0,+}\leqslant C_k^0(\gamma)<\infty$$

Proof. The case $\gamma=0$ is the mass-conservation of (7). To study the case $\gamma>0$ we let

$$\varphi: R_+\to R_+$$

denote a continuous function with

$$0\leqslant\varphi\leqslant 1,\qquad \varphi(x)=0\quad(x\leqslant 1),\qquad \varphi(x)=1\quad(x\geqslant 2)$$

For $0<\gamma\leqslant\beta$, an integration of (1) multiplied by

$$\varphi(j|v-v_1|)|v-v_1|^{-\gamma}$$

implies by (7)–(9), Lemma 4, and Lemma 5, that

$$D_t\int_{R^3} f(v_1,t)\varphi(j|v-v_1|)|v-v_1|^{-\gamma}\,dv_1$$
$$+C_k^0\int_{R^3} f(v_1,t)\varphi(j|v-v_1|)|v-v_1|^{-\gamma}\,dv_1$$
$$\leqslant D_t\int_{R^3} f(v_1,t)\varphi(j|v-v_1|)|v-v_1|^{-\gamma}\,dv_1$$
$$+\int_{R^3} Lf(v_1,t)f(v_1,t)\varphi(j|v-v_1|)|v-v_1|^{-\gamma}\,dv_1$$
$$=\int_{R^3} Jf(v_1,t)\varphi(j|v-v_1|)|v-v_1|^{-\gamma}\,dv_1\leqslant C_k\|f_0\|_{1,2}^2$$

And so by Lemma 3

$$\int_{R^3} f(v_1,t)\varphi(j|v-v_1|)|v-v_1|^{-\gamma}\,dv_1$$
$$\leqslant \max\left(\left\|\int_{R^3} f_0(v_1)|v-v_1|^{-\gamma}\,dv_1\right\|_{\infty,0}, C_k\|f_0\|_{1,2}^2/C_k^0\right)$$
$$\leqslant \max\left(C(\gamma)(\|f_0\|_{1,0}+\|f_0\|_{\infty,0}), C_k\|f_0\|_{1,2}^2/C_k^0\right)$$
$$\leqslant C_k^0(\gamma)<\infty \qquad (t>0)$$

For $0<\gamma\leqslant\beta$ the lemma follows in the limit $j\to\infty$. For $\beta\leqslant\gamma<2$ in the same way

$$D_t\int_{R^3} f(v_1,t)\varphi(j|v-v_1|)|v-v_1|^{-\gamma}\,dv_1$$
$$+C_k^0\int_{R^3} f(v_1,t)\varphi(j|v-v_1|)|v-v_1|^{-\gamma}\,dv_1$$
$$\leqslant C_k(\gamma)\|f_0\|_{1,0}\left\|\int_{R^3} f(v_1,t)|v-v_1|^{\beta-\gamma}\,dv_1\right\|_{\infty,0} \tag{22}$$

By the previous part of the proof, the last term is bounded for $t>0$, when $\beta\leqslant\gamma<2\beta$. Again by Lemma 3 this implies

$$\int_{R^3} f(v_1,t)\varphi(j|v-v_1|)|v-v_1|^{-\gamma}\,dv_1\leqslant C_k^0(\gamma)<\infty \qquad (t>0)$$

For $\beta\leqslant\gamma<2\beta$, the lemma follows in the limit $j\to\infty$. By induction the lemma follows for all γ with $0\leqslant\gamma<2$.

Lemma 10. Under (11) for a.e. plane $E_{v,\bar v}$

$$\int_{E_{v,\bar v}} f(v_1,t)\,dE_1<C_k^0 \qquad (t>0) \tag{23}$$

Proof. Define ϕ_j by (20). An integration of (1) multiplied by ϕ_j implies by (7)–(9) and Lemma 4 that

$$D_t\int_{R^3}\phi_j(v_1)f(v_1,t)\,dv_1+\bar C_k^0\int_{R^3}\phi_j(v_1)f(v_1,t)\,dv_1$$
$$\leqslant D_t\int_{R^3}\phi_j(v_1)f(v_1,t)\,dv_1+\int_{R^3}\phi_j(v_1)Lf(v_1,t)f(v_1,t)\,dv_1$$
$$=\int_{R^3}\phi_j(v_1)Jf(v_1,t)\,dv_1$$

By (7), Lemma 7, and Lemma 9 the right member can be estimated independently of j as

$$\int_{R^3}\phi_j(v_1)Jf(v_1,t)\,dv_1\leqslant C_k\|f\|_{1,0}\left\|\int_{R^3} f(v_2,t)|v_1-v_2|^{-\alpha+1}\,dv_2\right\|_{\infty,0}$$
$$\leqslant C_k\|f_0\|_{1,0}\bar C_k^0(\alpha-1)=C_k^0(\alpha-1)<\infty \qquad (t>0)$$

An application of Lemma 3 gives

$$\int_{R^3}\phi_j(v_1)f(v_1,t)\,dv_1 \leqslant \max\left(\int_{R^3}\phi_j(v_1)f_0(v_1)\,dv_1,\ C_k^0(\alpha-1)/\bar{C}_k^0\right)$$

For a.e. plane $E_{v,\bar{v}}$

$$\lim_{j\to\infty}\int_{R^3}\phi_j(v_1)f_0(v_1)\,dv_1 = \int_{E_{v,\bar{v}}} f_0(v_1)\,dv_1$$
$$\leqslant \|f_0\|_{\infty,s_2}\int_{R^2}\left[1+\left(y_1^2+y_2^2\right)^{1/2}\right]^{-s_2} dy_1\,dy_2$$

Hence

$$\int_{E_{v,\bar{v}}} f(v_1,t)\,dv_1 = \lim_{j\to\infty}\int_{R^3}\phi_j(v_1)f(v_1,t)\,dv_1 \leqslant C_k^0 < \infty \qquad (t>0)$$

Lemma 11. Set

$$\psi(v_1)=0 \quad \text{for} \quad |v_1|\leqslant|v|, \qquad \psi(v_1)=1 \quad \text{otherwise}$$

Under (11) for a.e. plane $E_{\tilde{v},\bar{v}}$

$$\int_{E_{\tilde{v},\bar{v}}}\psi(v_1)f(v_1,t)\,dE_1$$
$$\leqslant \max\left\{\int_{E_{\tilde{v},\bar{v}}}\psi(v_1)f_0(v_1)\,dE_1,\right.$$
$$\left. C_k^0(s_1')\left(1+\sup_{0\leqslant\tau\leqslant t}\|f(\cdot,\tau)\|_{\infty,0}\right)^2(1+|v|)^{-s_1'-1}\right\}$$

Here $s_1'=2$ if $s_2\leqslant 5$, and s_1' is only restricted by $s_1'<s_2-3$ if $s_2>5$.

Proof. Define ϕ_j by (20). An integration of (1) multiplied by $\phi_j\psi$ implies by (7)–(9) and Lemma 4 that

$$D_t\int_{R^3}\phi_j(v_1)\psi(v_1)f(v_1,t)\,dv_1 + \tilde{C}_k^0(1+|v|)^\beta\int_{R^3}\phi_j(v_1)\psi(v_1)f(v_1,t)\,dv_1$$
$$\leqslant D_t\int_{R^3}\phi_j(v_1)\psi(v_1)f(v_1,t)\,dv_1 + \int_{R^3}\phi_j(v_1)\psi(v_1)Lf(v_1,t)f(v_1,t)\,dv_1$$
$$= \int_{R^3}\phi_j(v_1)\psi(v_1)Jf(v_1,t)\,dv_1$$

By hypothesis $f_0\in L^1_{s_1'}$ and f satisfies (10) for s_1' if $s_1'>2$. So we can use Lemma 8 to estimate the right member,

$$\int_{R^3}\phi_j(v_1)\psi(v_1)Jf(v_1,t)\,dv_1 \leqslant \bar{C}_k^0(s_1')(1+\|f(\cdot,t)\|_{\infty,0})^2(1+|v|)^{-s_1'-\alpha+1}$$

Thus

$$D_t\int_{R^3}\phi_j(v_1)\psi(v_1)f(v_1,t)\,dv_1+\tilde{C}_k^0(1+|v|)^\beta\int_{R^3}\phi_j(v_1)\psi(v_1)f(v_1,t)\,dv_1$$
$$\leqslant \bar{C}_k^0(s_1')\big[1+\|f(\cdot,t)\|_{\infty,0}\big]^2(1+|v|)^{-s_1'-\alpha+1}$$

We recall that $\alpha+\beta=2$ and apply Lemma 3 to obtain

$$\int_{R^3}\phi_j(v_1)\psi(v_1)f(v_1,t)\,dv_1$$
$$\leqslant \max\Big(\int_{R^3}\phi_j(v_1)\psi(v_1)f_0(v_1)\,dv_1$$
$$\Big[1+\sup_{0\leqslant\tau\leqslant t}\|f(\cdot,\tau)\|_{\infty,0}\Big)^2(1+|v|)^{-s_1'-1}C_k^0(s_1')\Big]$$

The desired result follows in the limit $j\to\infty$ for a.e. plane $E_{\tilde{v},\bar{v}}$.

Lemma 12. Under (11) the following result holds:

$$\|f\|_{\infty,0,+}<C_k^0$$

Proof. By (1), (13), and Lemma 4

$$D_tf(v_1,t)+C_k^0(1+|v_1|)^\beta f(v_1,t)$$
$$\leqslant D_tf(v_1,t)+f(v_1,t)Lf(v_1,t)=Jf(v_1,t)$$
$$=\int_{R^3}f(v_1',t)\int_{E_{v_1,v_1',\epsilon}}f(v_2',t)h(\theta)\cos^{-2}\theta|v_1'-v_2'|^{-\alpha}\,dE_2'\,dv_1'$$
$$\leqslant C_k\int_{R^3}dv_1'\,f(v_1')|\,v_1'-v_1|^{-\alpha}\int_{E_{v_1,v_1'}}f(v_2')\,dE_2' \tag{24}$$

Using Lemma 9 and Lemma 10 we get

$$D_tf(v_1,t)+\bar{C}_k^0f(v_1,t)\leqslant C_k^0(\alpha)\qquad(t>0)$$

And so the desired result follows by Lemma 3.

Proof of Theorem 2. Given v_1, if $f(v)=0$ for $|v|\geqslant|v_1|/\sqrt{2}$, then

$$f(v_1')f(v_2')=0\qquad\big(v_2\in R^3,\,u\in B\big)$$

and so $Jf(v_1)=0$ if $f(v)=0$ for $|v|\geqslant|v_1|/\sqrt{2}$. To use this property of J we split f in the following way. Given v_1 we set

$$f=f_i+f_u(=f_{i,v_1}+f_{u,v_1})$$

with

$$f_i(v)\big[=f_{i,v_1}(v)\big]=f(v)\qquad\text{if}\quad|v|\leqslant|v_1|/\sqrt{2}\,,\qquad=0\quad\text{otherwise}$$

Then

$$Jf(v_1) = Jf_u(v_1) + Jf_if_u(v_1) + Jf_uf_i(v_1) + Jf_i(v_1)$$
$$= Jf_u(v_1) + Jf_if_u(v_1) + Jf_uf_i(v_1) \qquad (25)$$

From the representation (13) it follows that

$$Jf_if_u(v_1) = \int_{R^3} f_i(v_1')\int_{E_{v_1,v_1',\epsilon}} f_u(v_2')h(\theta)\cos^{-2}\theta\,|v_1'-v_2'|^{-\alpha}\,dE_2'\,dv_1'$$
$$\leqslant C_k\int_{R^3} f_i(v_1')|\,v_1-v_1'|^{-\alpha}\int_{E_{v_1,v_1'}} f_u(v_2')\,dE_2'\,dv_1' \qquad (26)$$

and analogously for Jf_uf_u. Also

$$Jf_uf_i(v_1) \leqslant C_k\int_{R^3}\int_0^{2\pi}\int_0^{\pi/2-\epsilon} f_u(v_1')f_i(v_2')\cos\theta\sin\theta\,|v_1'-v_2'|^{\beta}\,d\theta\,d\varphi\,dv_2$$
$$\leqslant C_k\int_{R^3}\int_0^{2\pi}\int_0^{\pi/2} f_u(v_1')f_i(v_2')\cos\theta\sin\theta\,|v_1'-v_2'|^{\beta}\,d\theta\,d\varphi\,dv_2$$
$$= C_k\int_{R^3}\int_0^{2\pi}\int_0^{\pi/2} f_i(v_1')f_u(v_2')\cos\theta\sin\theta\,|v_1'-v_2'|^{\beta}\,d\theta\,d\varphi\,dv_2$$
$$= C_k\int_{R^3} f_i(v_1')\int_{E_{v_1,v_1'}} f_u(v_2')\cos^{-1}\theta\,|v_1'-v_2'|^{-\alpha}\,dE_2'\,dv_1'$$
$$\leqslant C_k\int_{R^3} f_i(v_1')|\,v_1-v_1'|^{-\alpha}\int_{E_{v_1,v_1'}} f_u(v_2')\,dE_2'\,dv_1' \qquad (27)$$

By (25)–(27)

$$Jf(v_1) \leqslant C_k\int_{R^3} f(v_1')|v_1-v_1'|^{-\alpha}\int_{E_{v_1,v_1'}} f_u(v_2')\,dE_2'\,dv_1' \qquad (28)$$

By Lemma 12 $f \in L_2^{1,+}\cap L_0^{\infty}$, and so applying Lemma 6 and Lemma 11 to (28) we get

$$Jf(v_1,t) \leqslant \overline{C}_k^0(\bar{s}_2)(1+|v_1|)^{-c} \qquad (t>0) \qquad (29)$$

with

$$c = \min(s_2-2, \max(3,\bar{s}_2-2)) + \min(\alpha, s_1'(1-\alpha/3)+s_2'\alpha/3)$$

for any $\bar{s}_2 < s_2$ and $s_1' = 2$, $s_2' = 0$. But (29) inserted into (24) gives

$$D_tf(v_1,t) + \overline{C}_k^0(1+|v_1|)^{\beta}f(v_1,t) \leqslant Jf(v_1,t)$$
$$\leqslant \overline{C}_k^0(\bar{s}_2)(1+|v_1|)^{-c} \qquad (t>0)$$

By Lemma 3 this implies

$$\|f\|_{\infty,s_2'',+} \leqslant C_k^0(s_2'') < \infty \qquad \text{for} \quad s_2'' = c+\beta \qquad (30)$$

If $s_2 > 5$, then $s_2'' > 3$, and $f \in L_3^\infty$ with

$$\|f\|_{\infty,3,+} \leqslant C_k^0(3) < \infty$$

In this case iterating the argument once with $s_1' = 2$, $s_2' = 2$, and $5 \leqslant \bar{s}_2 < s_2$, we get (30) with $s_2'' = \bar{s}_2$, which completes the proof for the case $s_2 > 5$.

If $s_2 \leqslant 5$ and in (30) $s_2'' = s_2$, then the theorem holds. Otherwise

$$s_2 > s_2'' = s_2 - \alpha + 2(1 - \alpha/3)$$

and $f \in L_2^{1,+} \cap L^\infty_{s_2 - \alpha + 2(1-\alpha/3)}$. Repeating the same argument and using induction we either get (30) with $s_2'' = s_2$ after $\leqslant j$ steps, or

$$s_2 > s_2'' = [s_2 - \alpha + 2(1 - \alpha/3)] \sum_0^{j-1} (\alpha/3)^\nu$$

after j steps. But

$$\sum_0^\infty (\alpha/3)^\nu = (1 - \alpha/3)^{-1}$$

and so

$$[s_2 - \alpha + 2(1 - \alpha/3)] \sum_0^\infty (\alpha/3)^\nu > s_2 - \alpha + 2 > s_2$$

Thus after a finite number of steps (30) holds with

$$s_2'' = c + \beta = (s_2 - \alpha) + \alpha = s_2$$

This ends the proof of Theorem 2.

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